Exam 1 review

Tuesday, September 24, 2024 3:58 PM

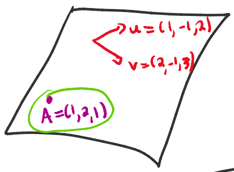
Exam_1_s...

Disclaimer : This is from last year under a different prof

$\rightsquigarrow ax+by+cz = \underbrace{\text{constant}}$

1. Consider the vectors $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and the point $A = (1, 2, 1)$.

1a. (15 pts) Find an equation of the plane that passes through A and is parallel to both $\mathbf{u}$ and $\mathbf{v}$.



Plane: Need (a) Point $P \in$ plane

(b) A Normal vector N

(b') Two vectors u, v parallel to plane

Then $N = u \times v$

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = \boxed{\begin{bmatrix} -1 \\ +1 \\ 1 \end{bmatrix} = N}$$

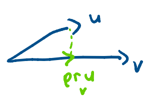
plane: $-1(x-1) + 1(y-2) + 1(z-1) = 0$

↳ open it up and move constant to the right

1b. (10 pts) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

$u = (1,-1,2)$

$v = (2,-1,3)$



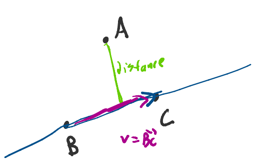
$$pr_v u = \frac{u \cdot v}{\|v\|^2} v$$

$$= \frac{2+1+6}{(\sqrt{4+1+9})^2} v = \frac{9}{14} v = \frac{9}{14}(2,-1,3)$$

$AB = (-1, -1, 3)$

2. Consider the points $A = (1, 2, -1)$, $B = (0, 1, 2)$, and $C = (2, 3, 1)$

2a. (15 pts) Find the distance from A to the line through B and C.



$\vec{BC} = (2, 2, -1) \parallel$ line

line: $r(t) = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}_{BC} t$

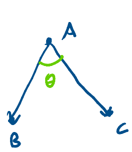
$$\text{dist}(A, \text{line}) = \frac{\|AB \times v\|}{\|v\|} = \frac{\|(-1,-1,3) \times (2,2,-1)\|}{\|(2,2,-1)\|} = \text{computations}$$

(pos. number)

$$\text{dist}(P, \text{line}) = \frac{\|PQ \times v\|}{\|v\|}$$ where Q is a pt on the line

$v =$ vector parallel to the line

2b. (10 pts) Find the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



$\vec{AB} = (-1, -1, 3) \rightsquigarrow \|AB\| = \sqrt{1+1+9} = \sqrt{11}$

$\vec{AC} = C - A = (1, 1, 2) \rightsquigarrow \|AC\| = \sqrt{1+1+4} = \sqrt{6}$

$u \cdot v = \|u\| \|v\| \cdot \cos(\theta)$

$AB \cdot AC = -1 - 1 + 6 = 4 = \|AB\| \cdot \|AC\| \cdot \cos(\theta)$

$\Rightarrow 4 = \sqrt{11} \cdot \sqrt{6} \cdot \cos(\theta) \Rightarrow \cos(\theta) = \frac{4}{\sqrt{66}} \rightsquigarrow \theta = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right) \rightsquigarrow$

$AB \cdot AC = -1-1+6 = 4 = \|AB\| \cdot \|AC\| \cdot \cos(\theta)$

$$\Rightarrow 4 = \sqrt{11} \cdot \sqrt{6} \cdot \cos(\theta) \Rightarrow \cos(\theta) = \frac{4}{\sqrt{66}} \rightsquigarrow \theta = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right) \rightsquigarrow$$

$r(t) = \text{position} \rightsquigarrow v(t) = r'(t) = \text{velocity} \rightsquigarrow a(t) = v'(t) = r''(t) = \text{acceleration}.$

3. The velocity of a particle in space at time t is given by $\mathbf{v}(t) = (t^2 - 2)\mathbf{i} - 2t\mathbf{j} - 2t\mathbf{k}$.

3a. (10 pts) Given the initial position $\mathbf{r}(0) = (0, 0, 1)$, find the position of the particle at time $t = 1$.

$$v(t) = \begin{bmatrix} t^2-2 \\ -2t \\ -2t \end{bmatrix} \rightsquigarrow r(t) = \int v(t) = \begin{bmatrix} \frac{t^3}{3} - 2t + c_1 \\ -t^2 + c_2 \\ -t^2 + c_3 \end{bmatrix}$$

$$r(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 1 \end{matrix} \rightsquigarrow r(t) = \begin{bmatrix} \frac{t^3}{3} - 2t \\ -t^2 \\ -t^2+1 \end{bmatrix}$$

$$r(1) = \begin{bmatrix} \frac{1}{3} - 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ -1 \\ 0 \end{bmatrix}$$

3b. (15 pts) Find the length of the curve traced by this particle from $t = 0$ to $t = 1$.

$$L = \int_0^1 \left\| \frac{dr}{dt} \right\| dt = \int_0^1 \|v(t)\| \, dt = \int_0^1 \left\| \begin{bmatrix} t^2-2 \\ -2t \\ -2t \end{bmatrix} \right\| dt$$

$\int \sqrt{1-t^2}$

$\cancel{\int \sqrt{4-5t^2}}$

$$= \int_0^1 \sqrt{(t^2-2)^2 + 4t^2 + 4t^2} \, dt$$

$$= \int_0^1 \sqrt{t^4 - 4t^2 + 4 + 8t^2}$$

$$= \int_0^1 \sqrt{t^4 + 4t^2 + 4} \, dt$$

$$= \int_0^1 \sqrt{(t^2+2)^2} \, dt$$

$$= \int_0^1 |\underbrace{t^2+2}_{>0}| \, dt = \int_0^1 (t^2+2) \, dt = \frac{t^3}{3} + 2t \Big|_0^1 = \frac{1}{3} + 2 = \frac{7}{3}$$

4a. (13 pts) Determine if the parametrization $\mathbf{r}(t) = (t - \sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k}$ with $0 \leq t \leq 4\pi$, is smooth, piece-wise smooth or neither.

Did it in class (piece-wise smooth)

~~4b.~~ (13 pts) Find the curvature of this curve at $t = \frac{\pi}{2}$. Simplify your answer.

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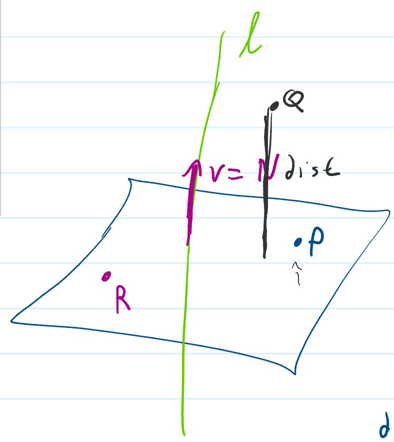

Distance to planes:

$\ell: \frac{x+5}{7} = \frac{y-11}{9} = \frac{z}{45}$

$P = (1, 0, 0)$

$Q = (2, 3, 4)$

Find the distance of the point Q to the plane that contains the point P and is perp. to the line ℓ.

$$\text{dist} = \frac{|\overset{\substack{\text{pt on plane} \\ \downarrow}}{QR} \cdot N|}{\|N\|}$$

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plane (a) point

(b) Normal vector N

(b') Two vectors u, v parallel to plane

$\ell: \frac{x+5}{7} = \frac{y-11}{9} = \frac{z}{45} = t \rightsquigarrow$

$x = -5 + 7t$

$y = 11 + 9t$

$z = 0 + 45t$

v = parallel to line

$z = 0 + 45t$

v = parallel to line

$v = N$ = perpend. to plane

$P = (1,0,0) \in$ plane $\leftarrow$

plane: $\boxed{7(x-1) + 9y + 45z = 0}$

pt $Q = (2,3,4)$

(Unecessary step: $7(2-1) + 9\cdot 3 + 45\cdot 4 \neq 0$ so Q is NOT on the plane)

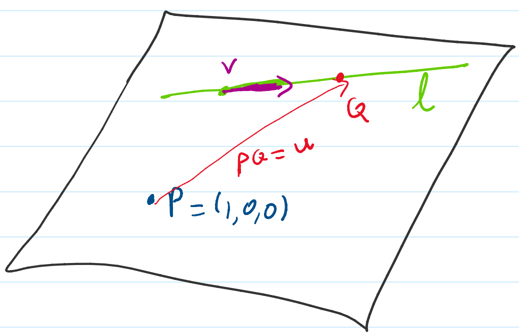
$$\text{dist}(Q, \text{plane}) = \frac{|PQ \cdot N|}{\|N\|}$$ where P is a pt on the plane $\rightsquigarrow PQ = Q - P = (1,3,4)$

$$= \frac{|(1,3,4)\cdot(7,9,45)|}{\|(7,9,45)\|} = \text{computations...}$$

$P = (1,0,0)$ on the plane

$\ell: r(t) = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}_{Q} + \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{v} t$ on the plane

Find the eqn of the plane



$\rightsquigarrow N = v \times \vec{PQ}$

$PQ = Q - P = (1,3,4)$
$v = (4,5,6)$
$\Big\} \Rightarrow N = (1,3,4) \times (4,5,6)$

Exam last fall

$$r(t) = \begin{bmatrix} t\sin(t) + \cos(t) \\ -t\cos(t) + \sin(t) \\ t^2 \end{bmatrix} \qquad 0 \le t \le 2$$

$$r'(t) = \begin{bmatrix} \cancel{\sin(t)} + t\cos(t) - \cancel{\sin(t)} \\ -\cancel{\cos(t)} + t\sin(t) + \cancel{\cos(t)} \\ 2t \end{bmatrix} = \begin{bmatrix} t\cos(t) \\ t\sin(t) \\ 2t \end{bmatrix}$$

$$L = \int_0^2 \|r'(t)\|\, dt = \int_0^2 \sqrt{t^2\cos^2 t + t^2\sin^2 t + 4t^2}$$

$$= \int_0^2 \sqrt{t^2(\cos^2(t) + \sin^2(t) + 4)}$$

$$= \int_0^2 \sqrt{5t^2} = \sqrt{5} \int_0^2 |t| = \sqrt{5}\int_0^2 t = \sqrt{5}\, \frac{t^2}{2}\Big|_0^2 = \boxed{2\sqrt{5}}$$

Q: $P = (1, 0, 0)$ Find → (i) Area of the triangle formed by P, Q, R

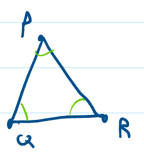
$Q = (1, 2, 1)$ (ii) Is the triangle a right angle triangle.

$R = (0, 0, 3)$

(ii) $PQ = (0, 2, 1)$ Check if $PQ \cdot PR = 0$?

$PR = (-1, 0, 3)$ $PQ \cdot QR = 0$?

$QR = (-1, -2, 2)$ $PR \cdot QR = 0$?

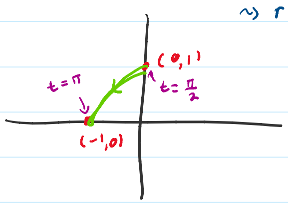


If one of them is zero, then it is a right angle

If none of them are, then it isn't.

Find a parametrization of a quarter circle that passes through $(0,1)$, $(-1,0)$ with center $(0,0)$

$$\rightsquigarrow r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \qquad t \in \left[\tfrac{\pi}{2}, \pi\right]$$



They will be on final exam.

$$a_T = \frac{v \cdot a}{\|v\|}$$

$a_N = \frac{\|v \times a\|}{\|v\|}$

Ex:

$r(t) = \begin{bmatrix} 2t \\ t^2 \\ \frac{1}{3}t^3 \end{bmatrix}$

Find the tangential component.

$v = r'(t) = \begin{bmatrix} 2 \\ 2t \\ t^2 \end{bmatrix}$

$a = v' = \begin{bmatrix} 0 \\ 2 \\ 2t \end{bmatrix}$

$a_T = \frac{v \cdot a}{\|v\|} = \frac{\begin{bmatrix} 2 \\ 2t \\ t^2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 2t \end{bmatrix}}{\|(2, 2t, t^2)\|} = \frac{4t + 4t^3}{\sqrt{4 + 4t^2 + t^4}}$